O.H: None tomorrow

Instead today 3PM—7PM (PST)

I. Lagrange multipliers

II. Making use of "conservative", interpreting $\int_C \vec{F}\cdot d\vec{r}$ intuitively

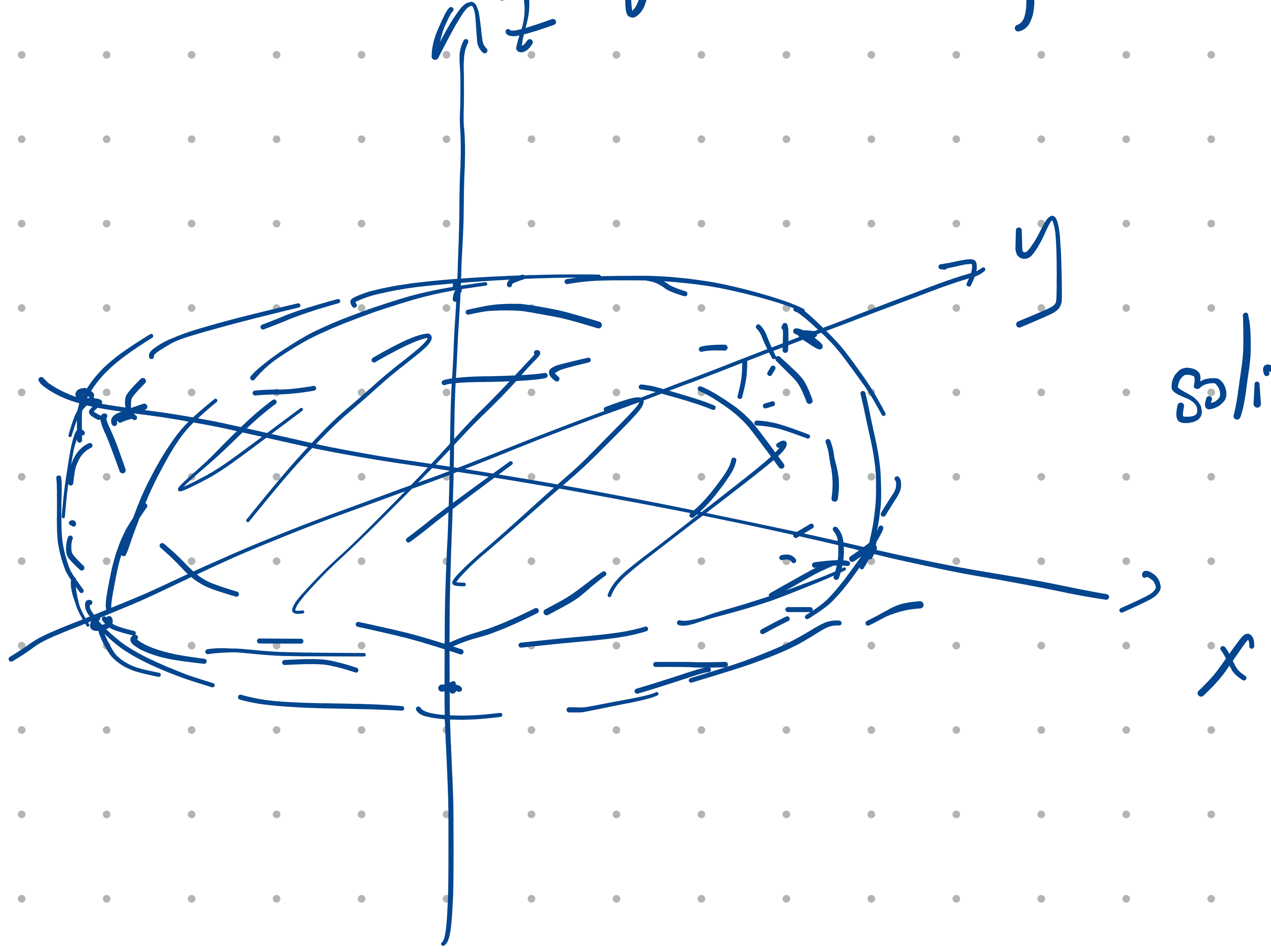
Quiz 9 #2

III. 15.9 change of var.

I. Find the abs. max & min of

$$f(x,y,z) = 2x^2+y^2+z^2$$

constrained to the region $x^2+y^2+2z^2 \leq 1$



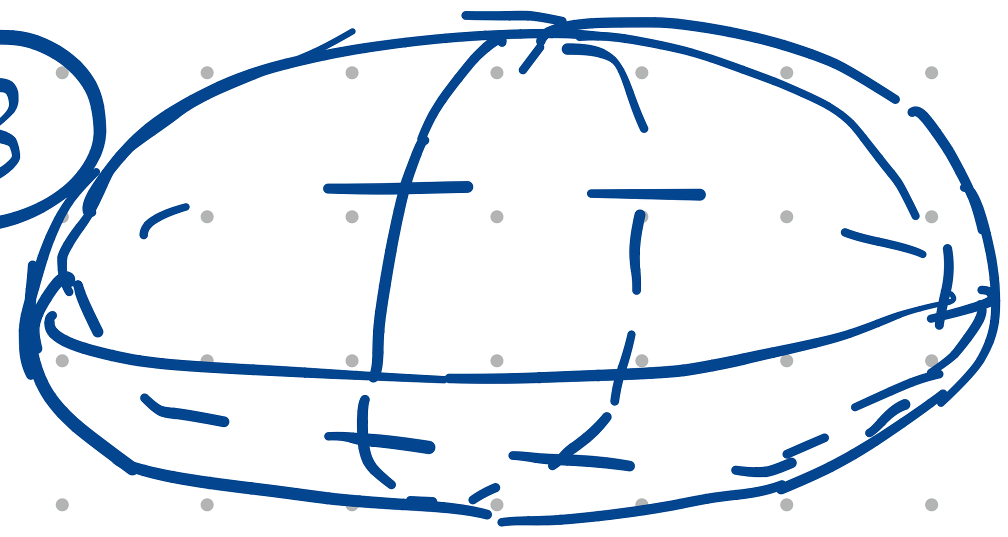
solid ellipsoid

0. Convince yourself extrema actually exist
(this problem already implies they do, but you could use EVT in this particular question)

1. Decompose the region: $x^2+y^2+2z^2 \leq 1$

(A) 

$x^2+y^2+2z^2 < 1$

(B) 

$x^2+y^2+2z^2 = 1$

2. For each region: find the candidates for extrema subject to any equality constraints and then verify they satisfy any inequality constraints

(A) No "equality constraints". Candidates are just when $\nabla f = \vec{0}$ i.e.

$$\langle 4x, 2y, 2z \rangle = \langle 0,0,0 \rangle$$

$$(x,y,z) = (0,0,0)$$

check: $0^2+0^2+2\cdot 0^2 = 0 < 1$ ✓

Ⓑ $\begin{cases} \underbrace{x^2+y^2+2z^2-1}_{g(x,y,z)}=0 \\ \nabla f = \lambda \nabla g \quad \text{i.e. } \langle 4x, 2y, 2z\rangle = \lambda\langle 2x, 2y, 4z\rangle \end{cases}$

$$2x = \lambda x$$
$$y = \lambda y$$
$$z = \lambda 2z$$

Examine $2x = \lambda x$. Case $x \neq 0$ or Case $x = 0$.

$$2 = \lambda$$

$$y = 2y \Rightarrow y = 0$$

$$z = 4z \Rightarrow z = 0$$

$$x^2 + 0^2 + 2\cdot 0^2 - 1 = 0 \Rightarrow x = \pm 1.$$

$(1,0,0)$

$(-1,0,0)$

In the case $x=0$:

$$y(1-\lambda)=0 \implies y=0 \text{ or } \lambda=1$$

Proceed in similar fashion...

Find $\left(0,0,\frac{1}{\sqrt{2}}\right),\ \left(0,0,-\frac{1}{\sqrt{2}}\right)$

$(0,1,0),\ (0,-1,0)$

3. Evaluate

Largest is abs max.
Smallest is abs min.

Ⓐ $f(0,0,0)$

Ⓑ $f(1,0,0)$

$f(-1,0,0)$

$f\left(0,0,\frac{1}{\sqrt{2}}\right)$

⋮ etc.

⚠ The points in this list are not necessarily local extrema.

Q: where does the 2nd derivative test fit in all this?

A: It doesn't. That's for classifying local behavior of a function $f(x,y)$ that is not (locally) constrained.

II. Quiz 9

1) a) Directly apply FTLI

b) Either recall that $\nabla(f^2) = 2f(\nabla f)$

or explicitly compute $f(\nabla f) = (x^2+y)\langle 2x, 1\rangle$

$$= \langle 2x^3 + 2xy, x^2+y\rangle$$

and find potential from there.

2) a) Compute $Q_x - P_y$

$$Q_x - P_y = (x+5e^y) - (x+5e^y)$$
$$= 0$$

Also domain of $\vec{F}$ is $\mathbb{R}^2$, which is simply connected

$\Longrightarrow$ $\vec{F}$ is conservative.

b) A lot of people parametrized C:

$r = \theta$

$$x = r\cos\theta = \boxed{\theta\cos\theta}$$
$$y = r\sin\theta = \boxed{\theta\sin\theta}$$

parametrization

$0 \leq \theta \leq 4\pi$

$$\int_0^{4\pi} \langle xy + 5e^y, \ldots \rangle \cdot \left\langle \frac{dx}{d\theta}, \frac{dy}{d\theta} \right\rangle d\theta$$

substitute in terms of θ

⚠ This is technically correct but misses the point of (a).

(a) tells us that $\vec{F}$ has a potential fn f

i.e. $\nabla f = \vec{F}$.

One thing we could try to do is find f:

$f_x(x,y) = xy + 5e^y$

$\Rightarrow f(x,y) = \frac{1}{2}x^2 y + 5xe^y + C(y)$

$\sin(y^2) + \frac{1}{2}x^2 + 5xe^y = f_y(x,y) = \frac{1}{2}x^2 + 5xe^y + C'(y)$

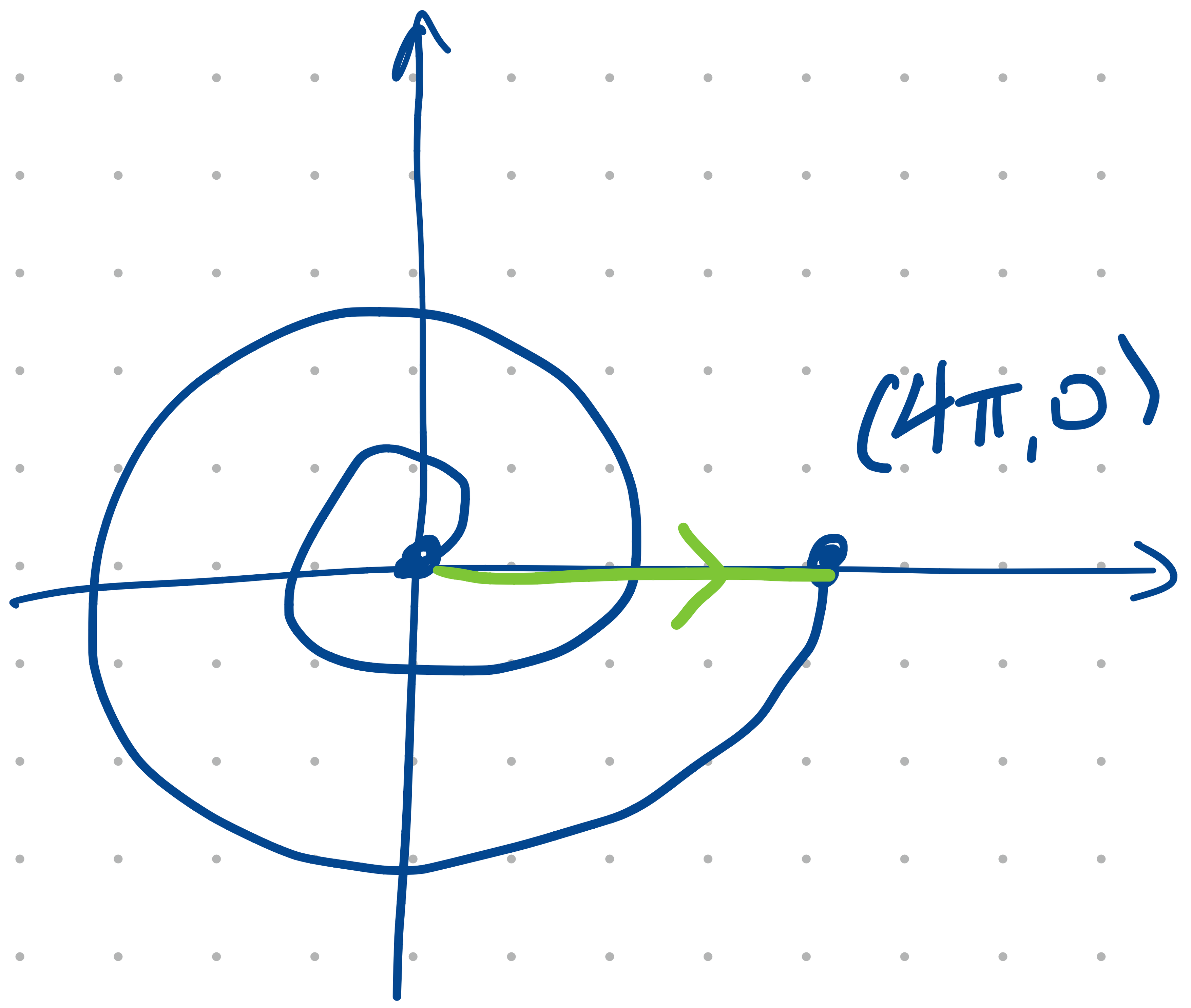
Let $C(y)$ be some antiderivative of $\sin(y^2)$.

then $f(x,y) = \frac{1}{2}x^2y + 5xe^y + C(y)$ is a pot. fn. for $\vec{F}$.

$$\int_C \vec{F} \cdot d\vec{r} = f(4\pi, 0) - f(0,0)$$

$$= 0 + \boxed{20\pi} + \cancel{C(0)}$$

$$-(0 + 0 + \cancel{C(0)})$$



Another way of interpreting (a) is path-independence,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$$

if C' is any curve that starts @ $(0,0)$
and ends @ $(4\pi, 0)$

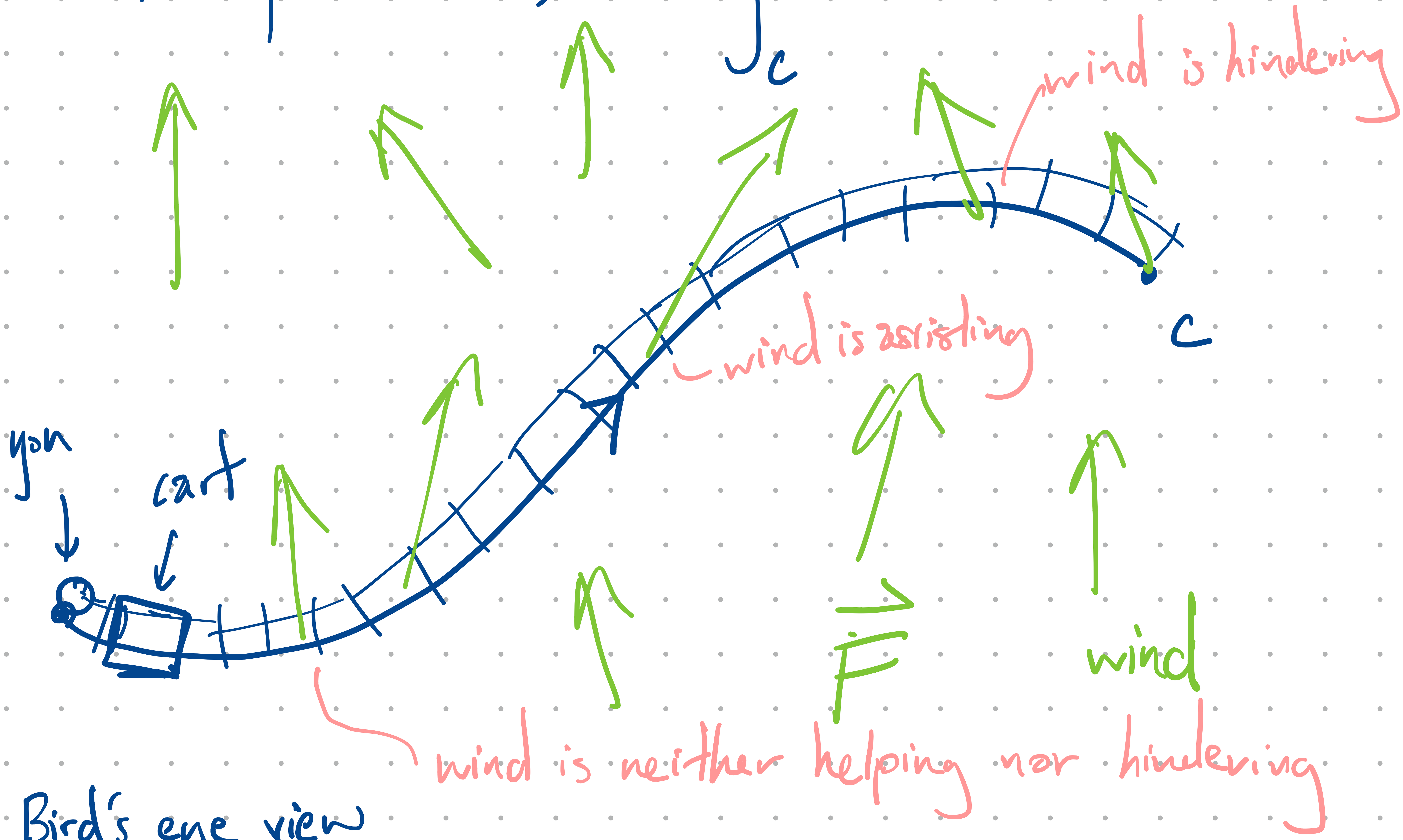
So let's use C' the straight line

$$\vec{r}(t) = \langle t, 0\rangle \qquad 0 \le t \le 4\pi$$

$$\int_{C'} \vec{F}\cdot d\vec{r} = \int \langle 0 + 5e^0, \ldots \rangle \cdot \langle 1, 0\rangle \, dt$$

$$= \int_0^{4\pi} 5 \, dt = \boxed{20\pi}$$

Interpretation of $\int_C \vec{F} \cdot d\vec{r}$



Bird's eye view

Q: How much does the wind assist (+) or hinder (−) you?